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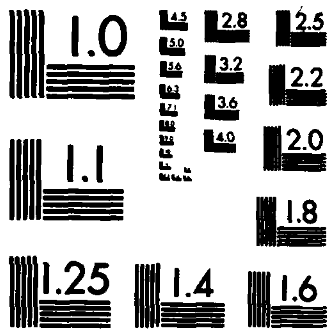

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AN ITERATIVE METHOD FOR SOLVING FINITE DIFFERENCE APPROXIMATIONS TO THE STOKES EQUATIONS

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ABSTRACT

A new iterative method is presented for solving finite difference equations which approximate the steady Stokes equations. The method is an extension of successive-over-relaxation and has two iteration parameters. Perturbation methods are used to analyze the iteration matrix. Sufficient conditions for the convergence of the iterative method are obtained and it is shown that many reasonable finite difference schemes for the Stokes equations satisfy these conditions. Computational examples are given to show the efficiency of the method.

AMS (MOS) Subject Classifications: 65F10, 65N20

Key Words: Successive-over-relaxation, Finite differences, Stokes Equations.

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SIGNIFICANCE AND EXPLANATION

The incompressible Navier-Stokes equations describe the flow of many common fluids. Thus effective numerical methods for solving these equations are very important for many scientific and engineering applications. In this paper a new algorithm is presented for solving finite difference equations for the linearized Navier-Stokes equations. The method is similar to successive-over-relaxation which is a widely used algorithm for solving elliptic difference equations. Numerical results showing the behavior of the method are presented. Other results appeared in an earlier report which discussed finite difference schemes for the incompressible Navier-Stokes equations. The method is efficient and easy to implement.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

AN ITERATIVE METHOD FOR SOLVING FINITE DIFFERENCE APPROXIMATIONS TO THE STOKES EQUATIONS

John C. Strikwerda

1. Introduction

In this paper we present and analyze a new iterative method for solving finite difference approximations to the steady Stokes equations. The method is a variant of successive-over-relaxation (S.O.R.) and has similarities to the method used by Chorin (1968) for the time-dependent Navier-Stokes equations. The method described here is called extended successive-over-relaxation (E.S.O.R.) and is useful for solving the nonlinear incompressible Navier-Stokes equations as well.

The Stokes equations are

$$\begin{aligned} -\nabla^2 \vec{u} + \vec{\nabla} p &= \vec{f} \\ \vec{\nabla} \cdot \vec{u} &= g \end{aligned} \qquad \text{in } \Omega \subset \mathbb{R}^k \tag{1.1}$$

and we take as boundary conditions

$$\vec{u} = \vec{b} \quad \text{on} \quad \partial\Omega \ .$$

The velocity $\vec{u}$ is a vector of dimension k and the pressure p is a scalar. The system (1.1) requires k boundary conditions which can be either of Dirichlet type, as given above, or some other type.

A commonly used method for solving (1.1) is to replace the second equation of (1.1), the divergence equation, by an elliptic equation for the pressure. The resulting finite difference approximation can be solved by iterative methods for elliptic equations, (e.g. Harlow and Welch (1965), Roache (1972)). The difficulty with this approach is that solutions to the derived system need not be solutions of the original system (1.1) (see

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Strikwerda (1983), and Greenspan et al. (1964)). Therefore, we consider only finite difference approximations to the system (1.1) in the form given there.

The finite difference approximation of (1.1) results in matrix equations of the form

$$\begin{pmatrix} A_h & G_h \\ D_h & 0 \end{pmatrix} \begin{pmatrix} u_h \\ p_h \end{pmatrix} = \begin{pmatrix} f_h \\ g_h \end{pmatrix} \tag{1.2}$$

where the matrices $-A_h$, G_h, and D_h result from finite difference approximations of the vector Laplacian, gradient, and divergence operators, respectively. A_h will be assumed to be a square n by n matrix, G_h an n by $m+1$ matrix, and D_h an $m+1$ by n matrix. We will denote the $n+m+1$ by $n+m+1$ matrix in (1.2) by Z_h. Systems of the form (1.2) also arise in solving the time-dependent Stokes and Navier-Stokes equations.

The equations (1.1) will not have a solution unless the integrability condition

$$\int_\Omega g = \int_{\partial\Omega} \vec{b} \cdot \vec{n} \tag{1.3}$$

is satisfied. Similarly, the matrix Z_h in (1.2) will, in general, be singular, and the system (1.2) will not have a solution unless the data are constrained to be orthogonal to the left null vectors of the matrix Z_h. We will assume that the rank deficiency of Z_h is only one, corresponding to the one integrability condition (1.3).

Rather than constraining the data to be orthogonal to the left null vector of Z_h, we prefer to consider both p_h and g_h as defined only up to arbitrary additive constants. That is, they are elements of the vector spaces $\mathbb{R}^{m+1}/\mathbb{R}$, where the quotient space is defined by $v_1 \equiv v_2$ if $v_1 - v_2$ has all components equal. Considered this way, Z_h is an $n+m$ by $n+m$ non-singular matrix. This approach to solving (1.1) is discussed in more detail in section 4 of Strikwerda (1983).

The E.S.O.R. algorithm discussed in this paper has been used to solve several test problems involving the Stokes equations (Strikwerda (1983)) and is being used by the author to solve for solutions of both the steady and time dependent Navier-Stokes equations. The method appears to be quite efficient. Use of E.S.O.R. as a pre-conditioner for a conjugate gradient algorithm is being investigated.

In the next section we will analyze a class of iterative methods for systems of the form (1.2) and in section 3 we will discuss how the methods behave as the mesh size varies. Several numerical examples which illustrate the utility of the method are given in section 4.

2. The Extended S.O.R. Method.

In this section we study a class of iterative methods to solve linear systems of the form

$$\begin{pmatrix} A & G \\ D & O \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \tag{2.1}$$

where A, G, and D are matrices of dimension $n \times n$, $n \times m$, and $m \times n$, respectively. We assume that the matrix

$$Z = \begin{pmatrix} A & G \\ D & O \end{pmatrix}$$

is non-singular, hence n is greater than or equal to m.

Systems of the form (2.1) often arise in the solution of constrained optimization problems, indeed, the solution of the Stokes equations may be regarded as the minimum of a quadratic functional under the constraint that the divergence of u is specified. If m, the number of constraints, is much smaller than n one can often eliminate m values of the unknown x using the second row of equations in (2.1) and so obtain a system which can be solved by standard methods, see e.g. Dyn and Ferguson (1982). If a cartesian grid is used on a rectangular region one can use a special technique developed by Amit, Hall, and Porsching (1980) to reduce the system to an n by n system. We, however, will consider the case where m is quite large and where there is not a natural or convenient way to reduce the system to one involving only n equations in n variables.

The iterative methods we will discuss are extensions of successive-over-relaxation (S.O.R.) as applied to the matrix A. We assume that A has been transformed so that

$$A = I - L - U$$

where L and U are strictly lower and upper triangular matrices, respectively. The S.O.R. iterative procedure applied to the system

$$Ax = a$$

is given by

$$x^{\nu+1} = x^{\nu} - \omega(x^{\nu} - Lx^{\nu+1} - Ux^{\nu} - a) \tag{2.2}$$

and we assume that this converges for ω satisfying $0 < \omega < \omega_0$ for some positive value of ω_0. For the basic theory of S.O.R. the reader is referred to Young (1971). Assuming that (2.2) converges is equivalent to assuming the following condition.

<u>Condition 2.1</u>

There is a positive constant ω_0 such that for $0 < \omega < \omega_0$ the roots of

$$\det\left(\frac{\lambda+\omega-1}{\omega} I - \lambda L - U\right) = 0 \tag{2.3}$$

satisfy $|\lambda| < 1$.

For the full system (2.1) we consider the extended successive-over-relaxation iterative procedure

$$\begin{aligned} x^{\nu+1} &= x^{\nu} - \omega(x^{\nu} - Lx^{\nu+1} - Ux^{\nu} + G_0 y^{\nu} + G_1 y^{\nu+1} - a) \\ y^{\nu+1} &= y^{\nu} - \gamma(D_0 x^{\nu} + D_1 x^{\nu+1} - b) \end{aligned} \tag{2.4}$$

where

$$G_0 + G_1 = G$$

and

$$D_0 + D_1 = D \quad .$$

The iterative parameters must be determined so that (2.4) is a convergent algorithm. The purpose of the analysis in this section is to find conditions under which (2.4) will converge.

Chorin (1968) used a scheme similar to (2.4) to solve for the velocity and pressure at each new time level for the time-dependent Navier-Stokes equations. In Chorin's method the matrix A is essentially the identity matrix and he set ω to be 1.0 and G_1 and D_0 were zero.

We rewrite (2.4) in matrix form as

$$X_1 w^{\nu+1} = X_0 w^{\nu} + c \tag{2.5}$$

where

$$w^{\nu} = \begin{pmatrix} x^{\nu} \\ y^{\nu} \end{pmatrix}, \quad c = \begin{pmatrix} a \\ \gamma b \end{pmatrix}$$

and

$$X_1 = \begin{pmatrix} \frac{1}{\omega} I - L & G_1 \\ \gamma D_1 & I \end{pmatrix},$$

$$X_0 = \begin{pmatrix} \frac{1-\omega}{\omega} I + U & -G_0 \\ -\gamma D_0 & I \end{pmatrix}.$$

The method (2.5) will converge if and only if all the eigenvalues of $X_1^{-1}X_0$ have absolute value less than one. The first result on the eigenvalues of $X_1^{-1}X_0$ is this lemma.

<u>Lemma 2.1</u>

For $\gamma = 0$ there are two classes of eigenvalues of $X_1^{-1}X_0$. There are n eigenvalues which are roots of (2.3) and m simple eigenvalues all equal to unity.

<u>Proof</u>

Let λ be an eigenvalue of $X_1^{-1}X_0$, then

$$0 = \det(\lambda I - X_1^{-1}X_0) = \det(\lambda X_1 - X_0)\det X_1^{-1} .$$

At $\gamma = 0$,

$$\det X_1 = \omega^{-n}$$

so X_1 is non-singular for small values of γ. We have

$$\lambda X_1 - X_0 = \begin{pmatrix} \frac{\lambda+\omega-1}{\omega} I - \lambda L - U & \lambda G_1 + G_0 \\ \gamma(\lambda D_1 + D_0) & (\lambda-1)I \end{pmatrix} \tag{2.6}$$

so at $\gamma = 0$ the eigenvalues of $X_1^{-1}X_0$ are either roots of (2.3) or are equal to unity. The eigenvalues equal to unity are easily seen to be simple because A is non-singular. In fact, for any $y \in \mathbb{R}^m$ the vector

$$\begin{pmatrix} -A^{-1}Gy \\ y \end{pmatrix}$$

is an eigenvector of $X_1^{-1}X_0$ at $\gamma = 0$. This proves Lemma 2.1.

The n eigenvalues of $X_1^{-1}X_0$ which are equal to the roots of (2.3) at $\gamma = 0$ will be called the S.O.R. eigenvalues of $X_1^{-1}X_0$.

We will now study the perturbation expansion of the eigenvalues of $X_1^{-1}X_0$ about $\gamma = 0$.

<u>Theorem 2.1</u>

Those eigenvalues $\lambda_j(\gamma)$ of $X_1^{-1}X_0$ such that $\lambda_j(0) = 1$ satisfy

$$\lambda_j(\gamma) = 1 - \eta_j\gamma + o(\gamma) \tag{2.7}$$

where η_j is an eigenvalue of $-DA^{-1}G$.

The proof of Theorem 2.1 depends on the following two lemmas.

<u>Lemma 2.2</u>

Let $T(\gamma)$ be an analytic matrix-valued function defined in a neighborhood of $\gamma = 0$. If λ_0 is a simple eigenvalue of $T(0)$ then the eigenvalues $\lambda_j(\gamma)$ of $T(\gamma)$ for which $\lambda_j(0) = \lambda_0$ satisfy

$$\lambda_j(\gamma) = \lambda_0 + \gamma\mu_j + o(\gamma) \quad ,$$

where μ_j is an eigenvalue of $T'(0)$.

Lemma 2.3

If a, b, c, and d are matrices of dimension $n \times n$, $n \times m$, $m \times n$, and $m \times m$, respectively, then

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det(a - bd^{-1}c) \det d, \quad \text{if} \quad \det d \neq 0$$

$$= \det(d - ca^{-1}b) \det a, \quad \text{if} \quad \det a \neq 0 \quad .$$

Proof of Lemma 2.2.

The proof easily follows from results of Kato (1966) but we give it here for completeness. For γ near zero we can find a non-singular analytic matrix valued function $P(\gamma)$ such that

$$P(\gamma)^{-1}T(\gamma)P(\gamma) = \tilde{T}(\gamma)$$

has a block form

$$\tilde{T}(\gamma) = \begin{pmatrix} \tilde{T}_0(\gamma) & 0 \\ 0 & \tilde{T}_1(\gamma) \end{pmatrix}$$

where $\tilde{T}(0) = \lambda_0 I$. We now consider only $\tilde{T}_0(\gamma)$. The eigenvalues of $\tilde{T}_0(\gamma)$ have expansions as Puiseux series

$$\lambda_j(\gamma) = \lambda_0 + \sum_{\ell=1}^{\infty} \lambda_{j\ell}\gamma^{\ell/p}$$

for some positive integer p. If $p = 1$, the result follows. Assume that $p > 1$, we will show that for $1 < \ell < p$, $\lambda_{j\ell}$ is zero.

Let

$$u_j(\gamma) = \sum_{\ell=0}^{\infty} u_{j\ell}\gamma^{\ell/p} \quad , \quad u_{j0} \neq 0$$

be the eigenvector corresponding to $\lambda_j(\gamma)$. ($u_j(\gamma)$ does not have a pole at $\gamma = 0$ since $\tilde{T}_0(0)$ is diagonal.) Since $\tilde{T}_0(\gamma) = \lambda_0 I + \gamma\tilde{T}_{01} + \gamma^2\tilde{T}_{02} + \cdots$ and

$$\tilde{T}_0(\gamma)u_j(\gamma) = u_j(\gamma)\lambda_j(\gamma) \quad ,$$

we see upon substituting the series for u_j and λ_j that $\lambda_{j\ell} = 0$ for $1 < \ell < p$. Moreover for $\ell = p$ we have

$$(\tilde{T}_1 - \lambda_{jp})u_{j0} = 0$$

which shows further that λ_{jp} is an eigenvalue of $\tilde{T}_1$ and u_{j0} is a corresponding eigenvector. This proves Lemma 2.2.

Proof of Lemma 2.3.

The result follows easily from the matrix factorization

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a - bd^{-1}c & bd^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ c & d \end{pmatrix}$$
$$= \begin{pmatrix} I & 0 \\ ca^{-1} & d - ca^{-1}b \end{pmatrix} \begin{pmatrix} a & b \\ 0 & I \end{pmatrix} .$$

Proof of Theorem 2.1.

Since the eigenvalues of $X_1^{-1}X_0$ which are equal to 1 for $\gamma = 0$ are simple, by Lemma 2.2 we have that

$$\lambda_j(\gamma) = 1 - \eta_j\gamma + o(\gamma) .$$

By Lemma 2.3 we have

$$0 = \det(\lambda_j X_1 - X_0) =$$

$$\det((\lambda_j - 1)I - \gamma(\lambda_j D_1 + D_0)(\frac{\lambda_j + \omega - 1}{\omega} I - \lambda_j L - U)^{-1}(\lambda_j G_1 + G_0))$$

$$\cdot \det(\frac{\lambda_j + \omega - 1}{\omega} I - \lambda_j L - U) .$$

Now at $\gamma = 0$, λ_j is 1, and $A = I - L - U$ is non-singular. So substituting the expansion for $\lambda_j(\gamma)$ we have

$$0 = \gamma^m \det(-\eta_j I - DA^{-1}G + o(\gamma)) .$$

Hence η_j is an eigenvalue of $-DA^{-1}G$ and this proves Theorem 2.1.

Since the iterative procedure (2.4) will be convergent only if the eigenvalues of $X_1^{-1}X_0$ are all less than one in magnitude, we see from Theorem 2.1 that for (2.4) to converge for positive values of γ we must assume that the following condition holds.

<u>Condition 2.2</u>.

All the eigenvalues of $-DA^{-1}G$ have positive real part.

Note that if all eigenvalues of $-DA^{-1}G$ have negative real part, then one can either multiply the last m equation of (2.1) by negative one (i.e. replace D by $-D$) or, equivalently take γ to be negative. If, however, some of the eigenvalues of $-DA^{-1}G$ have positive real part and others have negative real part then the method will not converge.

We now state the main result of this section.

<u>Theorem 2.3</u>

Conditions 2.1 and 2.2 are sufficient for the algorithm (2.4) to converge for γ and ω satisfying $0 < \gamma < \gamma_0$ and $0 < \omega < \omega_0$ for some positive values of γ_0 and ω_0. Furthermore, if A is non-singular then necessary conditions for (2.4) to converge for such γ and ω are that the roots of (2.3) satisfy $|\lambda| < 1$ and that the eigenvalues of $-DA^{-1}G$ have non-negative real part.

<u>Proof</u>

Consider the two groups of eigenvalues described in Lemma 2.1. By continuity of the eigenvalues as functions of the matrix elements we have that the S.O.R. eigenvalues satisfy $|\lambda| < 1$ for γ in some range $0 < \gamma < \gamma_0$ for $0 < \omega < \omega_0$. Then by Theorem 2.1 and Condition 2.2 the remaining eigenvalues of $X_1^{-1}X_0$ also satisfy $|\lambda| < 1$ for $0 < \gamma < \gamma_0$ for some γ_0. This proves the sufficiency condition.

If the algorithm (2.4) converges for $0 < \gamma < \gamma_0$ and $0 < \omega < \omega_0$, then for $\gamma = 0$ the eigenvalues of $X_1^{-1}X_0$ must satisfy $|\lambda| < 1$. Since A is non-singular the S.O.R. eigenvalues of $X_1^{-1}X_0$ are not equal to 1 for $\gamma = 0$. Thus the non-S.O.R. eigenvalues are simple and satisfy (2.7). The condition that $|\lambda| < 1$ for small positive γ implies that $\text{Re}\ \eta_j > 0$.

We conclude this section by obtaining expressions for the perturbation of the S.O.R. eigenvalues of the iteration matrix for the case where A is diagonalizable and has property A. Under these conditions the iteration matrix for S.O.R. applied to A has principle vectors of grade at most two, (Young (1971) p. 233-238).

Now let λ_{j0} be a simple S.O.R. eigenvalue for $X_1^{-1}X_0$ at $\gamma = 0$ and let

$$w_j = \begin{pmatrix} u_{j0} \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} u_{j1} \\ p_{j1} \end{pmatrix} + O(\gamma)$$

be the perturbation expansion of a right eigenvector of $X_1^{-1}X_0$ with eigenvalue

(2.8) $$\lambda_j = \lambda_{j0} + \gamma\lambda_{j1} + O(\gamma) \quad .$$

Let

$$\tilde{w}_j = \begin{pmatrix} v_{j0} \\ 0 \end{pmatrix}$$

be a left eigenvector at $\gamma = 0$ such that

$$(v_{j0}, u_{j0}) \neq 0 \quad .$$

Substituting the above expansions for the eigenvector and eigenvalue in the equation

$$(\tilde{w}_j, X_0 w_j) = \lambda_j(\tilde{w}_j, X_1 w_j)$$

we obtain

$$\lambda_{j1} = -\frac{(v_{j0}, (G_0 + \lambda_{j0}G_1)p_{j1})}{(v_{j0}, (\frac{1}{\omega} I - L)u_{j0})} \quad .$$

Since w_j is an eigenvector of $X_1^{-1}X_0$, we have

$$p_{j1} = (1 - \lambda_{j0})^{-1}(D_0 + \lambda_{j0}D_1)u_{j0}$$

and also

$$(1 - \lambda_{j0})(1/\omega\ I - L)u_{j0} = Au_{j0}$$

hence

(2.9)
$$\lambda_{j1} = -\frac{(v_{j0},(G_0+\lambda_{j0}G_1)(D_0+\lambda_{j0}D_1)u_{j0})}{(v_{j0},Au_{j0})} .$$

Similarly, if λ_{j1}^* is an S.O.R. eigenvalue of grade 2 at $\gamma = 0$, then we have

(2.10)
$$\lambda_j^* = \lambda_{j0}^* + \gamma^{1/2}\lambda_{j\,1/2}^* + O(\gamma)$$

where

(2.11)
$$(\lambda_{j\,1/2}^*)^2 = -\frac{(v_{j0},(G_0+\lambda_{j0}^*G_1)(D_0+\lambda_{j0}^*D_1)u_{j0})}{(v_0,Au_{1/2})}$$

and $u_{1/2}$ is defined by

$$\left(\frac{\lambda_{j0}^*+\omega-1}{\omega} I - \lambda_{j0}^*L - U\right)u_{1/2} + \left(\frac{1}{\omega} I-L\right)u_{j0} = 0 .$$

3. The Finite Difference Stokes Equations.

In this section we consider the application of the iterative method (2.4) to finite difference approximations of the Stokes equations. We first consider Conditions 2.1 and 2.2 to see if they are satisfied.

Since the matrix A arises from a discretization of the vector Laplacian, Condition 2.1 is very reasonable. If the finite difference grid is rectangular with uniform spacing and one uses the standard five-point discretization for the Laplacian, then Condition 2.1 is satisfied, Young (1971). In addition, A will be symmetric and have Property A, (Young (1971)).

Condition 2.2 will also be satisfied for appropriate difference schemes. The operator Q_h represented by $-D_h A_h^{-1} G_h$ is a finite difference approximation to the operator Q_0 defined on $L^2(\Omega)/\mathbb{R}$ as follows.

$$Q_0 p = q \quad \text{if}$$

$$q = \vec{\nabla} \cdot \vec{u}$$

where
$$\nabla^2 \vec{u} = \vec{\nabla} p \quad \text{in} \quad \Omega$$

with
$$\vec{u} = 0 \quad \text{on} \quad \partial\Omega \ .$$

Crozier (1974) has proved the following:

<u>Theorem 3.1</u>

If Ω is a connected, bounded domain in $\mathbb{R}^2$ with smooth boundary then the operator Q_0 is a bounded, positive definite operator on $L^2(\Omega)/\mathbb{R}$.

Therefore, if Q_h is a consistent approximation to Q_0 one can expect that the next condition holds.

Condition 3.1.

There are positive constants c_1 and c_2 such that for $0 < h < h_0$,

(3.1) $$c_1 \leq \text{Re}\ \eta_i \leq |\eta_i| \leq c_2$$

where the η_i are as in Theorem 2.1.

It is important to note that Condition 3.1 is not satisfied for all finite difference schemes. In particular, if one uses standard central difference to approximate both the gradient of the pressure and the divergence of the velocity, then numerical tests indicate that Condition 3.1 is not satisfied. In section 4, we will discuss difference schemes which satisfy Condition 3.1. Condition 3.1 is related to the regularity of the difference scheme (Bube and Strikwerda, 1983). Regular difference schemes are those whose solutions satisfy regularity estimates analogous to those satisfied by solutions of the differential equation.

We now consider the convergence behavior of the E.S.O.R. method. Suppose then that one has a finite difference approximation to the Stokes equations (1.2) for which the method (2.6) will converge for some positive values of γ and ω. One would like to know how to choose values of ω and γ so as to obtain a good rate of convergence for the method. We are unable to give rigorous estimates of the convergence rate, but we will now show that the following conjecture is quite plausible.

Conjecture 3.1

If the matrix A satisfies property A and Condition 3.1 is satisfied then there are positive constants c_0 and c_1 such that for $\omega = 2/(1+c_0 h)$ and $\gamma = c_1 h$ then

$$\rho(X_1^{-1} X_0) = 1 - Kh + O(h) \tag{3.2}$$

for some positive constant K.

Since the iteration matrix for S.O.R. applied to the discrete five point Laplacian satisfies a relation like (3.2), Conjecture 3.1, if true, shows that E.S.O.R. for the Stokes equations is roughly as efficient as S.O.R. for the five point Laplacian.

We argue for the Conjecture 3.1 as follows. If A satisfies Property A then for $\omega > \omega^*$, where ω^* is the optimal parameter for (2.2), and small positive values of γ, the S.O.R. eigenvalues satisfy (2.8) or (2.10) and have modulus $\omega-1 + O(\gamma)$. Consider first those λ_{j0} which are near $\omega - 1$, that is, λ_{j0} has the form

$$\lambda_{j0} \approx (\omega-1)e^{i\theta_j h}$$

where $\theta_j = O(1)$ and $\omega = 2 + O(h)$ as h tends to zero. If the finite difference approximations for the divergence and gradient are consistent then (2.9) can be approximated by

$$\lambda_{j1} \approx \frac{|D_h u_j|^2}{(u_j, \tilde{A} u_j)}$$

since the adjoint of the gradient is the negative of the divergence and where $\tilde{A}$ is the finite difference negative Laplacian, i.e. without the normalization which makes the diagonal elements unity. The discrete eigenvector u_j may be regarded as a representation of smooth vector function $\vec{u}$ and so the above ratio is approximated as

$$\lambda_{j1} \approx \frac{|\mathrm{div}\ \vec{u}|^2}{|\mathrm{grad}\ \vec{u}|^2}$$

and so λ_{j1} is $O(1)$ as h tends to zero.

On the other extreme where λ_{j0} is close to $-(\omega-1)$ the discrete eigenvector u_j is a very oscillatory function. Then we have

$$(D_0+\lambda_{j0}D_1)u_{j0} = O(h^{-1})$$

$$(G_0+\lambda_{j0}G_1) = O(h)$$

and

$$Au_{j0} = O(1)$$

and thus λ_{j1} is again $O(1)$ as h tends to zero.

For other values of λ_{j0} on the circle with radius $\omega - 1$, an argument similar to those above shows that λ_{j1} will be bounded as h tends to zero. For $\lambda_{j,1/2}$ as in (2.10) and (2.11), the conclusion is that $\lambda_{j,1/2}$ is proportional to $h^{1/2}$ as h tends to zero.

Therefore if γ is taken proportional to h in (2.8) and (2.10) and we assume that the terms which are $o(\gamma)$ in (2.8) and $\mathcal{O}(\gamma)$ in (2.10) become $o(h)$ and $\mathcal{O}(h)$, respectively, we then obtain (3.2). This last assumption is the one for which we have no theoretical justification. It is not unreasonable, however, and the numerical experiments confirm that Conjecture 3.1 is quite plausible.

4. Numerical Examples

In this section we present some numerical results of using the E.S.O.R. algorithm on a test problem. We consider the Stokes equations

$$\nabla^2 u - \frac{\partial p}{\partial x} = 0$$

$$(4.1) \qquad \nabla^2 v - \frac{\partial p}{\partial y} = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = g(x,y) = \cos \pi x \cos \pi y$$

on $0 < x,y < 1$ with u and v specified on the boundary. The exact solution is given by

$$u = (2\pi)^{-1} \sin \pi x \cos \pi y$$

$$v = (2\pi)^{-1} \cos \pi x \sin \pi y$$

$$p = \cos \pi x \cos \pi y \quad .$$

The discretization used a uniform grid with the same number of grid points in each direction. The second-order accurate five-point Laplacian was used to approximate the Laplacian. As mentioned in section 3 the choice of discretization for the gradient and divergence terms is crucial to satisfy Condition 3.1. We employed here the regularized centered differences (Strikwerda (1983)) given by

$$\frac{\partial p}{\partial x} \approx \delta_{x0} p - \frac{1}{6} h^2 \ \delta_{x-} \delta_{x+}^2 p \quad ,$$

$$\frac{\partial p}{\partial y} \approx \delta_{y0} p - \frac{1}{6} h^2 \ \delta_{y-} \delta_{y+}^2 p \quad ,$$

$$\frac{\partial u}{\partial x} \approx \delta_{x0} u - \frac{1}{6} h^2 \ \delta_{x+} \delta_{x-}^2 u \quad ,$$

$$\frac{\partial v}{\partial y} \approx \delta_{y0} v - \frac{1}{6} h^2 \ \delta_{y+} \delta_{y-}^2 v \quad ,$$

where h is the grid spacing and δ_{x0}, δ_{x+}, and δ_{x-} are the centered, forward, and backward difference operators in the x-direction. The operators

δ_{y0}, δ_{y+}, and δ_{y-} are defined similarly for the y-direction. To determine the pressure at boundary points a cubic interpolation was used, e.g.

$$p_{1j} = 3(p_{2j} - p_{3j}) - p_{4j} \quad .$$

In these tests the difference operators D_0 and G_1 were zero, i.e. $D_1 = D$, $G_0 = G$. This is perhaps the easiest scheme to implement of those considered here. Note that for the theory developed in section 2 it is necessary that the difference operators G_0 and G_1, together with G, annihilate constants so that they are defined on $\mathbb{R}^{m+1}/\mathbb{R}$. This scheme has been shown to be second-order accurate, Strikwerda (1983). Here we give results only on the efficiency of the solution algorithm, ESOR.

To support the conjecture 3.1, several runs were made where ω and γ were given by

$$\begin{aligned} \omega &= 2/(1 + c_0 h) \\ \gamma &= c_1 h \end{aligned} \tag{4.1}$$

for several values of c_0, c_1 and h. The iterative method was stopped when the quantities

$$\begin{aligned} &\|u^{n+1} - u^n\| \,/\, (\|u^{n+1}\|^2 + 1)^{1/2} \\ &\|v^{n+1} - v^n\| \,/\, (\|v^{n+1}\|^2 + 1)^{1/2} \\ &\|p^{n+1} - p^n\| \,/\, (\|p^{n+1}\|^2 + 1)^{1/2} \end{aligned} \tag{4.2}$$

were all less than 10^{-5}. The norms for u and v in (4.2) were the discrete L^2 norms, and the norm for p was the L^2 norm in the quotient space $\mathbb{R}^{m+1}/\mathbb{R}$. The computation of the norm in the quotient space will be discussed later. If Conjecture 3.1 were valid then the product of I, the number of iterations required for convergence, and h would tend to a limit as h tends to zero. To see this we observe that if the spectral radius is $1 - Kh$ then the number of iterations required for the relative change in successive iterates to be less than ε is determined by

$$(1 - Kh)^I \approx \varepsilon \quad .$$

This implies

$$hI \approx -(\log \varepsilon)/K \quad . \tag{4.3}$$

The results of these runs are shown in Table 1.

The results in Table 1 for $c_0 = 5.0$ and $c_1 = 5.0$ give excellent agreement with Conjecture 3.1. The variation in the values of $h \cdot I$ for other values of c_0 and c_1 can be explained by the presence of the $O(h)$ term in (3.2) and because the use of the norms is only an imperfect indicator of the spectral radius.

We now discuss the computation of the norms for the quotient spaces $\mathbb{R}^{m+1}/\mathbb{R}$. If X is a vector in $\mathbb{R}^{m+1}$ then the L^2 norm of its image in $\mathbb{R}^{m+1}/\mathbb{R}$ is

$$\|X\| = \left(\sum_{k=1}^{m+1} (X_k - \bar{X})^2\right)^{1/2}$$

where $\bar{X}$ is the mean of X, i.e.

$$\bar{X} = \frac{1}{m+1} \sum_{k=1}^{m+1} X_k \quad .$$

An efficient and accurate way to compute this norm is the algorithm of West (1979). This can be described as follows

Initialize $k = 1$

$M_1 = X_1$

$N_1 = 0$

repeat for $k = 2,\ldots,m+1$

$R = X_k - M_{k-1}$

$U = R/k$

$M_k = M_{k-1} + U$

$N_k = N_{k-1} + R * U * (k-1)$

Table 1

c_0	c_1	h	I	h•I
5.0	5.0	1/20	163	8.2
		1.30	246	8.2
		1/40	329	8.2
		1/60	497	8.3
		1/80	666	8.3
4.0	5.0	1.20	141	7.1
		1/30	219	7.3
		1/40	298	7.5
		1/60	467	7.8
		1/80	656	8.2
3.14	4.5	1/20	163	8.2
		1/30	254	8.5
		1/40	347	8.7
		1/60	582	9.7
4.0	6.0	1/20	132	6.6
		1/30	199	6.6
		1/40	267	6.7
		1/60	>500	---

finally

$$\overline{X} = M_{m+1}$$
$$\|X\| = (N_{m+1})^{1/2} .$$

This algorithm is stable as shown by Chan and Lewis (1979), and is very convenient. This is used to compute the norm of the pressure and the residual of the last equation in the Stokes equations.

The mean of the residual of the divergence equation is the quantity δ_h discussed in Strikwerda (1983), and for each of the cases reported here it was on the order of the truncation error.

The E.S.O.R. method described here was used to compute the solutions discussed in Strikwerda (1983) where accurate finite difference schemes for the Stokes equations are described. It is also being used in the computation of Taylor vortex solutions to the steady Navier-Stokes equations and in computations of solutions of the time-dependent Navier-Stokes equations. The results of this research will be reported when it is completed. It has been found to be a reliable algorithm, the main difficulty being the choice of ω and γ. Conjecture 3.1 provides a means of estimating good values of ω and γ. By finding good values of ω and γ for, say, $h = 1/10$, one can then use Conjecture 3.1 to obtain good estimates of ω and γ for smaller values of h.

5. Conclusion

The E.S.O.R. method has been rigorously analyzed for matrices of the form (2.1) with the main results stated in Theorem 2.3. For the particular case of difference approximations to the Stokes equations we have argued in section 3 that the assumptions required by Theorem 2.3 are reasonable for many finite difference schemes. The results of section 4 have confirmed that the E.S.O.R. method is indeed an efficient algorithm for the solution of the finite difference Stokes equations.

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A new iterative method is presented for solving finite difference equations which approximate the steady Stokes equations. The method is an extension of successive-over-relaxation and has two iteration parameters. Perturbation methods are used to analyze the iteration matrix. Sufficient conditions for the convergence of the iterative method are obtained and it is shown that many reasonable finite difference schemes for the Stokes equations satisfy these conditions. Computational examples are given to show the efficiency of the method.

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